Radboud University

Convergence of Peter–Weyl Truncations of Compact Quantum Groups

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Outline

1. Motivation: Spectral triples and spectral truncations
2. Background: Compact quantum metric spaces
3. Peter–Weyl truncations of compact quantum groups
4. Outlook: Fourier truncations

Motivation: Spectral triples and spectral truncations

Spectral triples

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Definition. A *spectral triple* is a triple $(\mathcal{A}, H, D)$ consisting of a Hilbert space H, a unital *-algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ and an essentially self-adjoint operator $D : H \supseteq \mathrm{Dom}(D) \to H$ such that

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Theorem [Connes '96-'13]. If $(\mathcal{A}, H, D)$ is a *commutative* unital spectral triple (+ extra structure and conditions), then

$$(\mathcal{A}, H, D) = (\mathrm{C}^\infty(M), \mathrm{L}^2(S_M), D_M).$$

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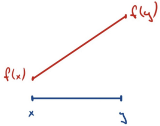

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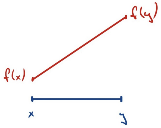

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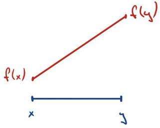

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Definition. Let $(\mathcal{A}, H, D)$ be a unital spectral triple. Then the *Monge–Kantorovich distance* on the state space $\mathcal{S}(\mathcal{A})$ is defined as

$$d^{\|[D,\cdot]\|}(\mu,\nu) := \sup_{\|[D,a]\|\leqslant 1} |\mu(a) - \nu(a)|.$$

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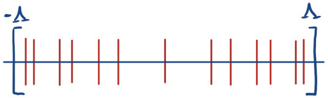

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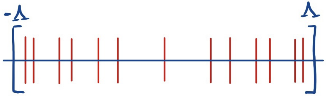

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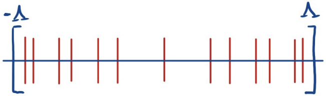

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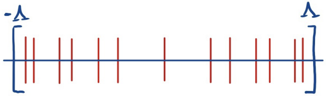

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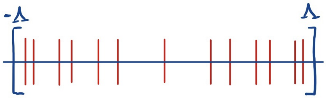

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Question. Do spectral truncations converge, as $P \to \mathbf{1}^H$?

Examples

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- *Spectral truncations* of $\mathbb{T}$ [van Suijlekom, Hekkelman].
- *Spectral truncations* of groups with polynomial growth [Toyota].
- *Peter–Weyl truncations* of compact groups [Gaudillot-Estrada–van Suijlekom].
- *Fourier truncations* of $\mathbb{T}$ [van Suijlekom].
- *Fourier truncations* of ergodic coactions of compact matrix quantum groups [Rieffel].

Background: Compact quantum metric spaces

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Definition. An (extended) seminorm $L_X : X \to [0, \infty]$ on an operator system X, such that $\mathrm{Dom}(L_X) := \{x \in X \mid L_X(x) < \infty\}$ is dense in X, $L_X(x^*) = L_X(x)$, for all $x \in X$, and $L_X(\mathbf{1}_X) = 0$ is called a *slip-norm*.

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Definition. Let (X, L_X), (Y, L_Y) be CQMS. A *morphism* is a ucp map $\Phi : X \to Y$ such that $L_Y(\Phi(x)) \leqslant CL_X(x)$.

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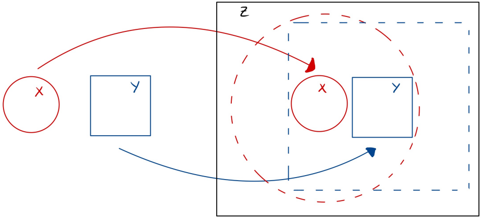

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Remark. $\mathrm{dist}_{\mathrm{GH}} \leqslant \mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}$. But the distances $\mathrm{dist}_{\mathrm{GH}}$ and $\mathrm{dist}^{\mathrm{q}}_{\mathrm{GH}}$ are not equivalent [KAAD–KYED '23].

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Theorem [Kerr]. Assume that the lip-norms L_X, L_Y are closed (i.e. $\mathrm{Dom}_1(L)$ closed in X_{sa}). Then $\mathrm{dist}^{\mathrm{s}}_{\mathrm{GH}}((X, L_X), (Y, L_Y)) = 0$ if and only if there is a bi-lip-isometric unital complete order isomorphism $X \to Y$.

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Theorem [Kerr–Li]. The set of isometry classes (appropriately defined using closures of lip-norms) of compact quantum metric spaces with $\mathrm{dist}^{\mathrm{s}}_{\mathrm{GH}}$ is a complete metric space.

Control of complete Gromov–Hausdorff distance

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Proposition [Rieffel '04, (Kerr '03), van Suijlekom '21, Kaad–Kyed '22]. Let (X, L_X) and (Y, L_Y) be CQMS, $\varepsilon > 0$. Suppose that there are lip-norm contractive morphisms $\tau : X \to Y$ and $\sigma : Y \to X$ such that

$$\|\sigma\tau(x) - x\| \leqslant \varepsilon L_X(x) \quad \text{and} \quad \|\tau\sigma(y) - y\| \leqslant \varepsilon L_Y(y).$$

Then $\mathrm{dist}^{\mathrm{s}}_{\mathrm{GH}}((X, L_X), (Y, L_Y)) \leqslant \varepsilon$.

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- $P_\Lambda \in \mathcal{B}(\mathrm{L}^2(G))$ orthogonal projection to $\mathrm{L}^2(G)_\Lambda := \bigoplus_{\gamma \in \Lambda} H_\gamma \otimes \overline{H_\gamma}$, for $\Lambda \subseteq \mathrm{Irr}(G)$ (finite).

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- The Lipschitz constant can be recovered using the actions λ, ρ:

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- Setting $\|\cdot\|_{\lambda,\rho} := \max\{\|\cdot\|_\lambda, \|\cdot\|_\rho\}$ makes sense on $\mathcal{B}(\mathrm{L}^2(G))$.
- The restriction of $\|\cdot\|_{\lambda,\rho}$ to $\mathrm{C}(G)^{(\Lambda)}$ defines a Lip-norm.

Theorem [Gaudillot-Estrada–van Suijlekom]. The net of metric spaces $(\mathcal{S}(\mathrm{C}(G)^{(\Lambda)}), d^{\|\cdot\|_{\lambda,\rho}})_{\Lambda \subseteq \mathrm{Irr}(G), |\Lambda| < \infty}$ converges to $(\mathcal{S}(\mathrm{C}(G)), d^{\|\cdot\|_{\lambda,\rho}})$ in Gromov–Hausdorff distance.

Compact quantum groups

Compact quantum groups

Definition [Woronowicz]. A *compact quantum group* is a separable unital C^*-algebra A (“$= \mathrm{C}(\mathbb{G})$”) together with a unital *-homomorphism $\Delta : A \to A \otimes A$ such that

- $(\Delta \otimes \mathbf{I}^A)\Delta = (\mathbf{I}^A \otimes \Delta)\Delta$,
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- Γ discrete group, $A := \mathrm{C}^*_\mathrm{r}(\Gamma) = \overline{\mathrm{L}^1(\Gamma)}^{\|\cdot\|_\mathrm{r}}$, $\Delta(\lambda_\gamma) = \lambda_\gamma \otimes \lambda_\gamma \in \mathrm{C}^*_\mathrm{r}(\Gamma) \otimes \mathrm{C}^*_\mathrm{r}(\Gamma) \cong \mathrm{C}^*_\mathrm{r}(\Gamma \times \Gamma)$.

Compact quantum groups

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- All our quantum groups are assumed *coamenable*, i.e.
  - the *counit* $\epsilon : A \to \mathbb{C}$ is a state,
    - $(\epsilon \otimes \mathrm{I}^A)\Delta(a) = (\mathrm{I}^A \otimes \epsilon)\Delta(a) = a$,
  - and the *Haar state* $h : A \to \mathbb{C}$ is faithful,
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  $\rightsquigarrow$ **NB.** $A \subseteq \mathcal{B}(H)$.
- The comultiplication $\Delta : A \to A \otimes A$ is implemented by the *multiplicative unitaries* $W, V \in \mathcal{B}(H \otimes H)$:

$$\Delta(a) = W(a \otimes \mathbf{1}_A)W^* = V(\mathbf{1}_A \otimes a)V^*$$

Peter–Weyl decomposition and truncations

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Theorem [“Peter–Weyl decomposition”]. The Hilbert space H and the multiplicative unitaries W, V decompose as $W = \bigoplus_{\gamma \in \mathrm{Irr}(\mathbb{G})} u^{\gamma}$, $V = \bigoplus_{\gamma \in \mathrm{Irr}(\mathbb{G})} u^{\overline{\gamma}}$ and

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Corollary. For the orthogonal projections $P : H \to H_{\gamma} \otimes \overline{H_{\gamma}}$, $\gamma \in \mathrm{Irr}(\mathbb{G})$, we have

$$[W, P_{\gamma} \otimes \mathbf{1}^{H}] = [V, \mathbf{1}^{H} \otimes P_{\gamma}] = 0.$$

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Definition. For $\Lambda \subseteq \mathrm{Irr}(\mathbb{G})$, $P_\Lambda := \bigoplus_{\gamma \in \Lambda} P_\gamma$, define

$$A^{(\Lambda)} := P_\Lambda A P_\Lambda \subseteq \mathcal{B}(P_\Lambda H).$$

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Theorem. The comultiplication $\Delta : A \to A \otimes A$ induces coactions $\alpha : A^{(\Lambda)} \to A^{(\Lambda)} \otimes A$, $\beta : A^{(\Lambda)} \to A \otimes A^{(\Lambda)}$:

$$(\tau \otimes \mathrm{I}^A)\Delta = \alpha\tau \text{ and } (\mathrm{I}^A \otimes \tau)\Delta = \beta\tau.$$

- α, β *cocommute*: $(\beta \otimes \mathrm{I}^A)\alpha = (\mathrm{I}^A \otimes \alpha)\beta$.
- α, β are *ergodic*: $(A^\Lambda)^\alpha = \mathbb{C}\mathbf{1}_{A^{(\Lambda)}}$, for the *fixed point set*

$$(A^\Lambda)^\alpha := \{x \in A^{(\Lambda)} \mid \alpha(x) = x \otimes \mathbf{1}_A\}.$$

Invariant lip-norms

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Let $L_A : A \to [0, \infty]$ be a lip-norm, which is *regular* (i.e. $\mathrm{Dom}(L_A) \supseteq \mathcal{O}(\mathbb{G})$) and *bi-invariant*, i.e.

$$L_A((\mathbf{1} \otimes \mu)\Delta(a)) \leqslant L_A(a)$$
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Example. $(\mathrm{C}(G), \mathrm{Lip}_d)$, where

- G compact group,
- d bi-invariant metric: $d(gh, gh') = d(hg, h'g) = d(h, h')$.

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Lemma. There is an induced slip-norm $L^{\alpha}_{A^{(\Lambda)}}$, which is invariant, i.e.:

$$L^{\alpha}_{A^{(\Lambda)}}((\mathbf{I}^{A^{(\Lambda)}} \otimes \mu)\alpha(x)) \leqslant L^{\alpha}_{A^{(\Lambda)}}(x),$$

for all $x \in A^{(\Lambda)}$, $\mu \in \mathcal{S}(A)$. Namely,

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Theorem. The slip-norm $L^{\alpha}_{A^{(\Lambda)}}$ is a regular *lip-norm* on $A^{(\Lambda)}$. Analogously for $L^{\beta}_{A^{(\Lambda)}}$ and $L^{\alpha,\beta}_{A^{(\Lambda)}} := \max\{L^{\alpha}_{A^{(\Lambda)}}, L^{\beta}_{A^{(\Lambda)}}\}$.

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Proof by ergodicity of the coactions on $A^{(\Lambda)}$ and a theorem of Li's. □

Peter–Weyl truncations as CQMS

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Corollary. The *Peter–Weyl truncation* $(A^{(\Lambda)}, L^{\alpha,\beta}_{A^{(\Lambda)}})$ is a compact quantum metric space with bi-invariant lip-norm $L^{\alpha,\beta}_{A^{(\Lambda)}}$.

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- Need lip-norm contractive ucp maps $\tau : A \leftrightarrow A^{(\Lambda)} : \sigma$, such that

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$$\sigma^{\phi}(x) := (\phi \otimes \mathbf{I}^A)\alpha(x),$$

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Focus on $\sigma^{\phi}\tau(a) = (\tau^*\phi \otimes \mathbf{I}^A)\alpha(a) = \tau^*\phi(a_{(0)})a_{(1)}$.

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Slice map lemma. For all $\mu, \nu \in \mathcal{S}(A)$, the following holds:

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Corollary. Let $\varepsilon > 0$. Then there is $\Lambda \subseteq \widehat{\mathbb{G}}$ finite, $\phi \in \mathcal{S}(A^{(\Lambda)})$ such that

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Proof.

- By Kadison function representation and Fubini for slice maps:

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□

Convergence of Peter–Weyl truncations

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Theorem [L]. Let $\mathbb{G}$ be a coamenable CQG and L_A a bi-invariant regular lip-norm on $A = \mathrm{C}(\mathbb{G})$. Then the Peter–Weyl truncations convergence in complete Gromov–Hausdorff distance, along the net of (finite) subsets $\Lambda \subseteq \mathrm{Irr}(\mathbb{G})$:

$$(A^{(\Lambda)}, L^{\alpha,\beta}_{A^{(\Lambda)}}) \xrightarrow{\Lambda} (A, L_A)$$

arXiv:2409.16698

Outlook: Fourier truncations

Fourier truncations

Fourier truncations

Definition. Let $\Lambda \subseteq \widehat{\mathbb{G}}$ such that $\overline{\Lambda} = \Lambda$ and $\mathbf{1} \in \Lambda$. Then the operator system $\mathrm{C}(\mathbb{G})_{(\Lambda)} := \bigoplus_{\gamma \in \Lambda} \mathbb{C}[\mathbb{G}]^{\gamma} \subseteq \mathrm{C}(\mathbb{G})$ is called a *Fourier truncation* of $\mathbb{G}$.

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Theorem [Rieffel]. Let $\mathbb{G}$ be a coamenable compact matrix quantum group and L a right-invariant regular lip-norm on $A = \mathrm{C}(\mathbb{G})$. Then the following sequence of Fourier truncations converges in quantum Gromov–Hausdorff distance:

$$\left(A_{(\Lambda^{\otimes n})}, L|_{A_{(\Lambda^{\otimes n})}}\right) \stackrel{n \to \infty}{\longrightarrow} (A, L)$$

Duality

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Proof.
The duality is given by:

$$((t_{i-j})_{i,j}, (\ldots, 0, f_{-N+1}, \ldots, f_{N-1}, 0, \ldots)) := \sum_{k=-N+1}^{N-1} t_k f_k$$

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NB. Fejér–Riesz lemma not available for many groups other than $\mathbb{T}^1$.

Propagation number

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Definition. Let X be an operator system. A C^*-extension is a C^*-algebra B together with a unital complete order embedding $\iota : X \hookrightarrow B$ such that $B = C^*(\iota(X))$. The *injective envelope* $(\mathrm{C}^*_{\mathrm{env}}(X), \iota)$ is the unique C^*-extension of X such that any ucp map $\phi : \mathrm{C}^*_{\mathrm{env}}(X) \to B$ is a unital complete order embedding if and only if $\phi \circ \iota$ is.

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Proposition [Connes–van Suijlekom, L–van Suijlekom]. For all $d \geqslant 1$, we have $\mathrm{C}^*_{\mathrm{env}}(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}) = \mathcal{B}(P_\Lambda \mathrm{L}^2(S_{\mathbb{T}^d}))$ and $\mathrm{prop}(\mathrm{C}(\mathbb{T}^d)^{(\Lambda)}) = 2$.

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Question. Peter–Weyl truncations of quantum homogeneous spaces?